数理解析研究所講究録
939巻 1996年 110-119

MINIMUM–TIME PROBLEM OF NONLINEAR CONTROL SYSTEM ON SEPARABLE REFLEXIVE BANACH SPACES

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1. INTRODUCTION

We consider the nonlinear control system N_0 given by

$$\begin{cases} \dot{x}(t) = A_0x(t) + Fx(t) + B_0u(t), & t \geq 0 \\ x(0) = x_0 \end{cases}$$

in the separable reflexive Banach space X_0. Along with N_0, the sequence $\{N_n\}, n = 1, 2, \cdots$, of perturbed equations

$$\begin{cases} \dot{x}_n(t) = A_nx_n(t) + Fx_n(t) + B_nu_n(t), & t \geq 0 \\ x_n(0) = x_{0,n} \end{cases}$$

in the separable reflexive Banach spaces X_n is considered with the mild solution

$$x_n(t) = S_n(t)x_{0,n} + \int_0^t S_n(t-s)\{Fx_n(s) + B_nu_n(s)\}ds$$

for every $u_n(\cdot) \in Y_n$, where Y_n is a control space and $B_n \in \mathfrak{L}(Y_n, X_n)$. The operator A_n and the nonlinear operator F are assumed to satisfy that A_n generates a strongly continuous semigroup of bounded linear operator $S_n(t), t \geq 0$, on X_n and $A_n + F$ is strongly dissipative. $u_n(\cdot)$ is a locally summable function.

Linear case ($F = 0$) of above systems in Hilbert space have been treated by Carija ([3]).

In this paper, we consider the case where $B_n = I_n$ (the identity operator in X_n) and we are to prove the existence of minimal time for the nonlinear system N_0 which steers initial value x_0 to the target x_1 and to give conditions for the convergence of the sequence of minimal times for the nonlinear approximate system N_n on X_n, $n = 1, 2, \cdots$, to the minimal time for the original system N_0 on X_0.

2. Minimum-Time Problem

We consider the nonlinear control systems

$$\dot{x}_n(t) = A_n x_n(t) + F x_n(t) + u_n(t) \quad t \geq 0 \tag{1}$$

in the separable reflexive Banach spaces X_n, $n = 0, 1, 2, \cdots$, with the mild solutions

$$x_n(t) = S_n(t)x_{0,n} + \int_0^t S_n(t-s)\{Fx_n(t) + u_n(s)\}ds. \tag{2}$$

For each $n \geq 0$, the set U_{ad}^n of admissible controls is defined by

$$U_{ad}^n = \{\text{strongly measurable function } u_n(\cdot);\ u_n(t) \in Y_n, \ \|u_n(t)\| \leq 1, \text{ a.e.}\}.$$

For $n \geq 0$, define

$$R_n(t) = \{(x_{0,n}, x_{1,n}) \in X_n \times X_n; x_n(0) = x_{0,n},\ x_n(t) = x_{1,n}, \ \text{for some } u_n \in U_{ad}^n\}$$

where $x_n(t)$ is given by (2). Define also

$$R_n = \cup_{t>0} R_n(t)$$

and the minimal-time function $T_n : R_n \to R^1$,

$$T_n(x_{0,n}, x_{1,n}) = inf\{t : (x_{0,n}, x_{1,n}) \in R_n(t)\}.$$

We now list the assumptions which will be in effect throughout this paper:

(A1) there exist $M > 0$ and $\omega \geq 0$, such that, for $n = 0, 1, 2, \ldots$ and $t \geq 0$,

$$\|S_n(t)\| \leq Me^{\omega t}$$

where M and ω are independent of n,

(A2) $S_n(t)$ is compact,

(A3) $S_n(t) \to S_0(t)$, uniformly for t in bounded intervals,

(A4) $S_n^*(t) \to S_0^*(t)$, uniformly for t in bounded intervals.

(F1) the nonlinear function F is Lipschitz continuous:
there exists a constant c, such that

$$\|Fx_n - Fy_n\| \leq c\|x_n - y_n\|, \qquad x_n, y_n \in X_n,$$

(F2) F has a linear growth rate on X_n; these exists a constant $k > 0$, such that

$$\|Fx_n\| \leq k(1 + \|x_n\|).$$

THEOREM 1. *If $x_0 \in X_0$ and $x_1 \in D(A_0)$, such that*

$$\|(A_0 + F)x_1\| + \omega\|x_0 - x_1\| < 1 \qquad \text{for } \omega > 0 \tag{3}$$

holds, then there exists $u \in U^0_{ad}$ which steers x_0 to x_1 in a time T_0 satisfying

$$T_0 \leq \omega^{-1} \log \{ \frac{1 - \|(A_0 + F)x_1\|}{1 - \|(A_0 + F)x_1\| - \omega\|x_0 - x_1\|} \} \tag{4}$$

Proof. Consider the nonlinear equation

$$\begin{cases} \dot{x}(t) &= A_0 x(t) + Fx(t) - sign(x(t) - x_1) \\ x(0) &= x_0 \in D(A_0) \end{cases} \tag{5}$$

where

$$sign(y) = y/\|y\|, \qquad y \neq 0,$$
$$sign(0) = \{z \in X : \|z\| \leq 1\}.$$

Thus,multiplying equation (5) with $x(t) - x_1 \neq 0$, using the dissipative of $A_0 + F$, multiplying by $e^{-2\omega t}$ and then integrating 0 to t(see [4]). We have

$$e^{-2\omega t}\|x(t) - x_1\|^2$$
$$\leq \|x_0 - x_1\|^2 - 2\int_0^t e^{-2\omega s}(1 - \|(A_0 + F)x_1\|)\|x(s) - x_1\|ds.$$

By Gronwall's inequality,

$$e^{-\omega t}\|x(t) - x_1\|$$
$$\leq \|x_0 - x_1\| - \int_0^t e^{-\omega s}(1 - \|(A_0 + F)x_1\|)ds$$

and then,

$$e^{-\omega t}\|x(t) - x_1\|$$
$$\leq \|x_0 - x_1\| + \omega^{-1}(1 - \|(A_0 + F)x_1\|)e^{-\omega t} - \omega^{-1}(1 - \|(A_0 + F)x_1\|).$$

Thus

$$\|x(t) - x_1\|$$
$$\leq e^{\omega t}\|x_0 - x_1\| - \omega^{-1}(1 - \|(A_0 + F)x_1\|)e^{\omega t} + \omega^{-1}(1 - \|(A_0 + F)x_1\|).$$

Let $x(t) \to x_1$, then

$$e^{\omega T_0}\{(1 - \|(A_0 + F)x_1\|) - \omega\|x_0 - x_1\|\} \leq 1 - \|(A_0 + F)x_1\|).$$

We also

$$e^{\omega T_0} \leq \frac{1 - \|(A_0 + F)x_1\|}{1 - \|(A_0 + F)x_1\| - \omega\|x_0 - x_1\|}.$$

Hence

$$T_0 \leq \omega^{-1} \log\left\{\frac{1 - \|(A_0 + F)x_1\|}{1 - \|(A_0 + F)x_1\| - \omega\|x_0 - x_1\|}\right\}.$$

We will assume that a mild solution exists for every $u_n(\cdot) \in L^p_{Y_n}$ and clearly, because of (F1), is unique.

LEMMA 1. *Let conditions (A1)-(A4), (F1)-(F2) and*

(B1) $$x_{0,n} \to x_0, \ x_{1,n} \to x_1$$

be satisfied. If

(6) $$(x_{0,n}, \ x_{1,n}) \in R_n(t_n)$$

(7) $$t_n \to T, \ u_n \rightharpoonup u \qquad \text{as } n \to \infty$$

then $(x_0, x_1) \in R_0(T)$.

Proof. Condition (6) implies that there exists $u_n \in U^n_{ad}$ such that

$$x_{1,n} = S_n(t_n)x_{0,n} + \int_0^{t_n} S_n(t_n - s)\{Fx_{1,n}(s) + u_n(s)\}ds$$

By (7), there exists T_0 such that

$$t_n \leq T_0, \ n \geq 1.$$

For every $n \geq 1$ and every $t \in [0, T_0]$, we have

$$\begin{aligned}
&\|x_{1,n} - x_1\| \\
&= \|S_n(t_n)x_{0,n} + \int_0^{t_n} S_n(t_n - s)\{Fx_{1,n}(s) + u_n(s)\}ds \\
&\quad - S_0(T)x_0 - \int_0^T S_0(T - s)\{Fx_1(s) + u(s)\}ds\| \\
&\leq \|S_n(t_n)x_{0,n} - S_0(T)x_0\| \\
&\quad + \|\int_0^{t_n} S_n(t_n - s)u_n(s)ds - \int_0^T S_0(T - s)u(s)ds\|
\end{aligned}$$

$$+ \| \int_0^{t_n} S_n(t_n - s)Fx_{1,n}(s)ds - \int_0^T S_0(T-s)Fx_1(s)ds\|$$
$$= I + II + III.$$

First, it is not hard to show that

$$S_n(t_n)x_{0,n} \;\to\; S_0(T)x_0.$$

$$\begin{aligned} II \;\le\; & \| \int_T^{t_n} S_n(t_n - s)u_n(s)ds\| \\ & + \| \int_0^T (S_n(t_n - s)u_n(s) - S_0(T-s)u_n(s))ds\| \\ & + \| \int_0^T (S_0(T-s)u_n(s) - S_0(T-s)u(s))ds\|. \end{aligned}$$

The first term converges to zero by (A1). The second term converges to zero by (A4). For the moment, let us concentrate on the third term. From the Hahn-Banach theorem, we know that we can find $x_n^* \;\in\; B_1^* =$ dual unit ball such that

$$\begin{aligned} & |(\int_0^T S_0(T-s)(u_n(s) - u(s))ds,\; x_n^*)| \\ & = \| \int_0^T (S_0(T-s)u_n(s) - S_0(T-s)u(s))ds\| \\ \Rightarrow\quad & | \int_0^T (u_n(s) - u(s),\; S_0^*(T-s)x_n^*)ds| \\ & = \| \int_0^T (S_0(T-s)u_n(s) - S_0(T-s)u(s))ds\|. \end{aligned}$$

From Schauder's theorem, we know that, for $T \;>\; s$, $S_0^*(T-s)$ is compact. By Alaoglu's theorem, we know that B_1^* is w-compact. So by passing to subsequence if necessary, we may assume that $x_n^* \;\to\; x^* \in B_1^*$. Hence, $S_0^*(T-s)x_n^* \;\to\; z^*(t)$. Since $u_n \;\rightharpoonup\; u$,

$$\begin{aligned} & | \int_0^t (u_n(s) - u(s),\; S_0^*(T-s)x_n^*)ds| \;\to\; 0 \\ \Rightarrow\quad & \| \int_0^t S_0(T-s)(u_n(s) - u(s))ds\| \;\to\; 0, \end{aligned}$$

as $n \;\to\; \infty$.

$$\begin{aligned} III \;\le\; & \| \int_T^{t_n} S_n(t_n - s)Fx_{1,n}(s)ds\| \\ & + \| \int_0^T (S_n(t_n - s)Fx_{1,n}(s) - S_n(t_n - s)Fx_1(s))ds\| \\ & + \| \int_0^T (S_n(t_n - s)Fx_1(s) - S_0(T-s)Fx_1(s))ds\|. \end{aligned}$$

First and third term converges to zero. Let

$$r_n(t) \;=\; I + II + [\text{First and third term of } III] \;\to\; 0,$$

as $n \;\to\; \infty$. We have

$$\begin{aligned}
&\|x_{1,n}(t) - x(t)\| \\
&\le r_n(t) + \| \int_0^T (S_n(t_n - s)Fx_{1,n}(s) - S_n(t_n - s)Fx_1(s))ds\| \\
&\le r_n(t) + MK \int_0^T e^{\omega(t_n - s)} \|x_{1,n}(s) - x_1(s)\| ds.
\end{aligned}$$

Using Gronwall's inequality, we get that

$$\begin{aligned}
&\|x_{1,n}(t) - x_1(t)\| \\
&\le r_n(t) + MK \int_0^T r_n(s) e^{\omega(t_n - s)} \exp(\int_0^T e^{\omega(t_n - \tau)} d\tau) ds.
\end{aligned}$$

But note that $\int_0^T e^{\omega(t_n - \tau)} d\tau \;\le\; R$. So we have

$$\|x_{1,n}(t) - x_1(t)\| \;\le\; r_n(t) + MK \exp(R) \int_0^T e^{\omega(t_n - s)} r_n(s) ds.$$

Recall that, for all $t \;\in\; [0, T]$, $r_n(t) \;\to\; 0$. So using the dominated convergence theorem, we get that $r_n(\cdot) \;\to\; 0$. Since

$$\int_0^T e^{\omega(t_n - s)} r_n(s) ds \;\le\; M' \|r_n\|,$$

$$\lim_{n \to \infty} \int_0^T e^{\omega(t_n - s)} r_n(s) ds \;\to\; 0.$$

Therefore

$$\|x_{1,n}(t) \;-\; x_1(t)\| \;\to\; 0$$

as $n \;\to\; \infty$. Hence $x_{1,n}(t) \;\to\; x_1(t)$, as $n \to \infty$, for all $t \in [0, T]$.

LEMMA 2. *Assume (A1)-(A4), (B1),*

(B2) $$x_1 \;\in\; D(A_0), \;\; \|(A_n + F)x_1\| \;<\; 1;$$

(B3) $$x_{1,n} \;\in\; D(A_n), \;\; A_n x_{1,n} \;\to\; A_0 x_1.$$

If $(x_0, x_1) \in R_0(t)$, *then there exists a sequence* $\{\gamma_n\}$, *convergent to zero, such that*

$$(x_{0,n},\ x_{1,n}) \in R_n(t+\gamma_n) \tag{8}$$

for n *sufficiently large.*

Proof. First of all, we prove the following assertion: if $y_n \to x_1$, then there exists a sequence $\{\gamma_n\}$ convergent to zero such that

$$(y_n,\ x_{1,n}) \in R_n(\gamma_n) \tag{9}$$

for n sufficiently large. Indeed, since

$$\|(A_0+F)x_1\| < 1,$$

there exists a positive integer n_1 such that, for $n \geq n_1$, we have

$$\|(A_n+F)x_{1,n}\| \leq c_1 < 1.$$

Furthermore, since $y_n \to x_1$, we may conclude that

$$\|(A_n+F)x_{1,n}\| + \omega\|y_n - x_{1,n}\| < 1, \quad n \geq n_1.$$

So, by Theorem 1, $x_{1,n}$ can be reached from y_n in a time T_n which satisfies

$$T_n \leq \omega^{-1}\log\Big\{\frac{1-\|(A_n+F)x_{1,n}\|}{1-\|(A_n+F)x_{1,n}\|-\omega\|y_n-x_{1,n}\|}\Big\}.$$

Taking $\gamma_n = T_n$, we obtain (9) and $\gamma_n \to 0$ as $n \to \infty$ as claimed. Since $(x_0, x_1) \in R_0(t)$, there exists $u \in U_{ad}^0$ such that

$$x_1 = S_0(t)x_0 + \int_0^t S_0(t-s)\{Fx_1(s)+u(s)\}ds. \tag{10}$$

Denoting

$$y_n = S_n(t)x_{0,n} + \int_0^t S_n(t-s)\{Fy_n(s)+u_n(s)\}ds. \tag{11}$$

$$
\begin{aligned}
&\|y_n(t) - x_1(t)\| \\
&\leq \|S_n(t)x_{0,n} - S_0(t)x_0\| \\
&\quad + \|\int_0^t S_n(t-s)(Fy_n(s) + u(s))ds - \int_0^t S_0(t-s)(Fx_1(s) + u(s))ds\| \\
&\leq \|S_n(t)x_{0,n} - S_0(t)x_0\| \\
&\quad + \int_0^t \|S_n(t-s) - S_0(t-s)\| \|u_n(s)\| ds \\
&\quad + \int_0^t \|S_0(t-s)(u_n(s) - u(s))\| ds \\
&\quad + \int_0^t \|S_n(t-s) - S_0(t-s)\| \|Fx_1(s)\| ds \\
&\quad + \int_0^t \|S_n(t-s)\| \|Fy_n(s) - Fx_1(s)\| ds \\
&= J1 + J2 + J3 + J4 + J5.
\end{aligned}
$$

$J1, J2$ and $J4$ are converge to zero as $n \to \infty$. By same method of Lemma 2, J3 is converge to zero as $n \to \infty$. Let $k_n(t) = J1 + J2 + J3 + J4 \to 0$ as $n \to \infty$. We have

$$
\begin{aligned}
&\|y_n(t) - x_1(t)\| \\
&\leq k_n(t) + \int_0^t \|S_n(t-s)\| \|Fy_n(s) - Fx_1(s)\| ds \\
&\leq k_n(t) + MK \int_0^t e^{\omega(t-s)} \|y_n(s) - x_1(s)\| ds.
\end{aligned}
$$

Using Gronwall's inequality, we get that

$$
\begin{aligned}
&\|y_n(t) - x_1(t)\| \\
&\leq k_n(t) + MK \int_0^t k_n(s) e^{\omega(t-s)} \exp(\int_0^t e^{\omega(t-\tau)} d\tau) ds.
\end{aligned}
$$

But note that $\int_0^t e^{\omega(t-\tau)} d\tau \leq R'$. So, we have

$$
\|y_n(t) - x_1(t)\| \leq k_n(t) + MK \exp(R') \int_0^t e^{\omega(t-s)} k_n(s) ds.
$$

Recall that, for any $t > 0$, $k_n(t) \to 0$. So using the dominated convergence theorem, we get that $k_n(\cdot) \to 0$. Since

$$
\int_0^t e^{\omega(t-s)} k_n(s) ds \leq M'' \|k_n\|,
$$

where M'' is constant,

$$\lim_{n\to\infty}\int_0^t e^{\omega(t-s)}k_n(s)ds \ \to \ 0.$$

Therefore $\|y_n(t) - x_1(t)\| \ \to \ 0$ as $n \ \to \ \infty$. Hence $y_n \ \to \ x_1$, as $n \ \to \ \infty$, for any $t > 0$. Therefore, (9) holds for y_n defined by (11).

Finally, by (9) and (11), we obtain (8), thereby completing our proofs.

THEOREM 2. *Under conditions (A1)-(A4), (F1)-(F2), assume that (B1)-(B3) and* $(x_0,\ x_1) \ \in \ R_0$. *Then, the following results hold:*

(a) $(x_{0,n},\ x_{1,n}) \ \in \ R_n$, *for n sufficiently large.*

(b) $\lim_{n\to\infty} T_n(x_{0,n},\ x_{1,n}) \ = \ T_0(x_0,\ x_1)$.

Proof. By Lemma 2, there exists a subsequence of $\{T_n(x_{0,n},\ x_{1,n})\}$, denoted by $\{T_{n'}\}$, which converges, say to T'. Using once again Lemma 2, with $t \ = \ T_0(x_0,\ x_1)$, we obtain

$$T_{n'} \ \leq \ T_0(x_0,\ x_1) \ + \ \gamma_{n'}.$$

Hence, we obtain

$$T' \ \leq \ T_0(x_0,\ x_1).$$

Finally, using Lemma 1, we may infer that

$$T_0(x_0,\ x_1) \ \leq \ T',$$

and thus we obtain

$$T_0(x_0,\ x_1) \ = \ T'.$$

Since the last equality can be obtained for all convergent subsequence, the proof is complete.

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